

A Note on Measure-Expansive Diffeomorphisms

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Abstract

In this note we prove that a homeomorphism is countably-expansive if and only if it is measure-expansive. This result is applied for showing that the C^1 -interior of the sets of expansive, measure-expansive and continuum-wise expansive C^1 -diffeomorphisms coincide.

1 Introduction

The phenomenon of expansiveness occurs when the trajectories of nearby points are separated by the dynamical system. The first research who considered expansivity in dynamical systems was by Utz [24]. There, he defined the notion of unstable homeomorphism. An extensive literature related to properties of expansiveness can be found in [1, 2, 4–10, 12, 13, 15, 17–19, 21, 23, 25–27].

If $f: M \rightarrow M$ is a homeomorphism of a compact metric space (M, dist) and if $\delta > 0$ we define

$$\Gamma_\delta(x) = \{y \in M : \text{dist}(f^n(x), f^n(y)) \leq \delta \text{ for all } n \in \mathbb{Z}\}.$$

Let us recall some definitions that can be found for example in [16]. We say that f is *expansive* if there is $\delta > 0$ such that $\Gamma_\delta(x) = \{x\}$ for all $x \in M$. Given a Borel probability measure μ on M we say that f is μ -*expansive* if there is $\delta > 0$ such that for all $x \in M$ it holds that $\mu(\Gamma_\delta(x)) = 0$. In this case we also say that μ is an *expansive measure* for f . We say that f is *measure-expansive* if it is μ -expansive for every non-atomic Borel probability measure μ . Recall that μ is non-atomic if $\mu(\{x\}) = 0$ for all $x \in M$. The corresponding concepts for flows have been considered in [3]. Moreover, we say that f is *countably-expansive* if there is $\delta > 0$ such that for all $x \in M$ the set $\Gamma_\delta(x)$ is countable.

In [16] it is proved that the following statements are equivalent:

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1. f is countably-expansive,
2. every non-atomic Borel probability measure of M is expansive with a common expansive constant.

Moreover, they make the following question: are there measure-expansive homeomorphisms of compact metric space which are not countably-expansive? We give a negative answer in Theorem 2.1.

We next study robust expansiveness of C^1 -diffeomorphisms of a smooth manifold. For a fixed manifold M , we denote by \mathcal{E} the set of all expansive diffeomorphisms of M . In order to state our next result let us recall more definitions. We say that $C \subset M$ is a *continuum* if it is compact and connected. A *trivial continuum* (or *singleton*) is a continuum with only one point. Recall from [11, 12] that f is *continuum-wise expansive* (or *cw-expansive*) if there is $\delta > 0$ such that if $C \subset M$ is a non-trivial continuum then there is $n \in \mathbb{Z}$ such that $\text{diam}(f^n(C)) > \delta$. Denote by \mathcal{CE} the set of all cw-expansive diffeomorphisms and by \mathcal{PE} the set of all measure-expansive diffeomorphisms of M . We denote by $\text{int } A$ the C^1 -interior of a set A of C^1 -diffeomorphisms of M . In [14] R. Mañé proved that the C^1 -interior of the set of expansive diffeomorphisms coincides with the set of quasi-Anosov diffeomorphisms. See [14] for the definitions and the proof. This result was later extended for cw-expansive homeomorphisms in [20] proving that $\text{int } \mathcal{E} = \text{int } \mathcal{CE}$. Recently, it was proved in [22] that $\text{int } \mathcal{E} = \text{int } \mathcal{PE}$. In Theorem 2.4 we give a new proof of the cited result from [22] based on Theorem 2.1 and [20].

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2 Proofs of the results

Our first result holds for a homeomorphism $f: M \rightarrow M$ of a compact metric space (M, dist) .

Theorem 2.1. *The following statements are equivalent:*

1. f is countably-expansive,
2. f is measure-expansive.

Proof. Direct. Let $\delta > 0$ be such that for all $x \in M$ it holds that $\Gamma_\delta(x)$ is countable. Let μ be a non-atomic Borel probability measure. Since μ is non-atomic, by σ -additivity we have that $\mu(\Gamma_\delta(x)) = 0$. Therefore, f is measure-expansive.

Converse. Arguing by contradiction, we assume that f is measure-expansive but there are sequences $\delta_n \rightarrow 0$ and $x_n \in M$ such that $\Gamma_{\delta_n}(x_n)$ is uncountable for each $n \geq 1$. As in [16], for each $n \geq 1$ consider a non-atomic Borel probability

measure μ_n such that $\mu_n(\Gamma_{\delta_n}(x_n)) = 1$. Consider the Borel probability measure μ defined for a Borel set $A \subset M$ as

$$\mu(A) = \sum_{n=1}^{\infty} \frac{\mu_n(A)}{2^n}.$$

Since every μ_n is non-atomic, we have that μ is non-atomic too. Thus, since f is measure-expansive, there is $\delta > 0$ such that $\mu(\Gamma_{\delta}(x)) = 0$ for all $x \in M$. Since $\delta_n \rightarrow 0$ we can take $\delta_n < \delta$. Then

$$\mu(\Gamma_{\delta}(x_n)) \geq \mu(\Gamma_{\delta_n}(x_n)) \geq \frac{\mu_n(\Gamma_{\delta_n}(x_n))}{2^n} > 0.$$

This contradiction proves the theorem. \square

For the proof of Theorem 2.4 we recall some known facts.

Lemma 2.2. *The following statements are equivalent:*

1. f is cw-expansive,
2. there is $\delta > 0$ such that for all $x \in M$ it holds that $\Gamma_{\delta}(x)$ contains no non-trivial continua.

Proof. For the direct part, consider a cw-expansive constant $\varepsilon > 0$ and take $\delta = \varepsilon/2$. If $C \subset \Gamma_{\delta}(x)$ is a connected component then $\text{diam}(f^n(C)) \leq 2\delta$ for all $n \in \mathbb{Z}$. Since $\varepsilon = 2\delta$ is a cw-expansive constant, we conclude that C is a singleton. Then, every continuum contained in $\Gamma_{\delta}(x)$ is trivial for all $x \in M$.

In order to prove the converse we consider $\delta > 0$ such that every $\Gamma_{\delta}(x)$ has no non-trivial continua. Let us show that δ is a cw-expansive constant. Suppose that $C \subset M$ is a continuum and $\text{diam}(f^n(C)) \leq \delta$ for all $n \in \mathbb{Z}$. Given $x \in C$ we have that for all $y \in C$ it holds that $\text{dist}(f^n(x), f^n(y)) \leq \delta$. Therefore, $y \in \Gamma_{\delta}(x)$. Since y is arbitrary, we have that $C \subset \Gamma_{\delta}(x)$. By hypothesis, we have that $\Gamma_{\delta}(x)$ contains no non-trivial continuum, therefore, C is a singleton. \square

Lemma 2.3. *The following implications hold:*

$$\text{expansive} \Rightarrow \text{countably-expansive} \Rightarrow \text{cw-expansive}.$$

Proof. The first implication is obvious because singletons are countable sets. The second one holds because every non-trivial continuum is uncountable. Therefore, if $\Gamma_{\delta}(x)$ is countable, it cannot contain any non-trivial continuum. By Lemma 2.2 we have that f is cw-expansive. \square

Now assume that f is a C^1 -diffeomorphism of a compact smooth manifold M and recall the definitions from the introduction.

Theorem 2.4 ([22]). *The following equality holds:*

$$\text{int } \mathcal{E} = \text{int } \mathcal{PE}.$$

Proof. By definition of the sets \mathcal{E} , \mathcal{PE} and using Theorem 2.1 and Lemma 2.3 we have that

$$\text{int } \mathcal{E} \subset \text{int } \mathcal{PE} \subset \text{int } \mathcal{CE}.$$

Finally, by Theorem 1 in [20] we have that

$$\text{int } \mathcal{CE} \subset \text{int } \mathcal{E}.$$

This finishes the proof. \square

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